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The smallest Laplacian spectral radius of graphs with a given clique number☆

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ABSTRACT

The Laplacian spectral radius of a graph G is the largest eigenvalue of its Laplacian matrix. In this paper, the first three smallest values of the Laplacian spectral radii among all connected graphs with clique number ω are obtained.

1. Introduction

Let $G = (V, E)$ be an undirected finite simple graph. By choosing a fixed ordering $v_1, v_2, \ldots, v_n$ of the set V, let $d(v_i)$ denote the degree of $v_i \in V(G)$, $(i = 1, 2, \ldots, n)$, and $D(G) = diag(d(v_1), \ldots, d(v_n))$ the diagonal matrix of vertex degrees. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of the graph G, where $A(G)$ denotes the adjacency matrix of G. It is well known that $L(G)$ is positive semidefinite and singular. Moreover, if G is connected then $L(G)$ is irreducible. Denote its eigenvalues by

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0,$$

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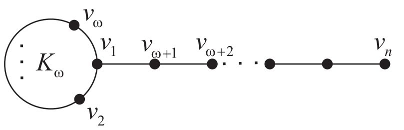

Fig. 1. Kite graph $PK_{n-\omega,\omega}$.

which are always enumerated in non-increasing order and repeated according to their multiplicity. The largest eigenvalue of $L(G)$ is called the *Laplacian spectral radius* of the graph G and is denoted by $\mu(G)$. It has been extensively used in investigating combinatorial optimization (see [14]) and communication networks (see [18]) etc. Recent discoveries indicate that the Laplacian spectral radius of trees plays an important role in the theory of the photoelectron spectra of saturated hydrocarbons (see [7,8]).

The characteristic polynomial of a square matrix B is denoted by $\Phi(B) = \det(xI - B)$. In particular, if $B = L(G)$, we write $\Phi(L(G))$ by $\Phi(G)$ (the Laplacian characteristic polynomial of G) for convenience. Readers are referred to [1] for undefined terms and notation.

Let X be an eigenvector of G corresponding to $\mu(G)$. Let x_i (or x_{v_i}) denote the entry of X corresponding to the vertex v_i of G. Such vertex labeling is sometimes called "valuation" [12]. If X is a unit eigenvector of G corresponding $\mu(G)$, then we have

$$\mu(G) = \max_{\substack{Y\in\mathbb{R}^n \\ ||Y||=1}} Y^T L(G) Y = X^T L(G) X = \sum_{\substack{v_iv_j\in E \\ 1\leq i<j\leq n}} (x_i - x_j)^2. \tag{1.1}$$

In [16,20], the first seven and 11 smallest Laplacian spectral radius are obtained among all trees, respectively. Furthermore, Shen et al. in [17] obtained the first 14 connected graphs with the smallest Laplacian spectral radii among all the connected graphs of order $n \geq 17$, which reach the first nine smallest Laplacian spectral radii among all connected graphs of order n when n is even (where some graphs are juxtaposed) and reach the first eight smallest spectral radii when n is odd (where some graphs are juxtaposed); the authors in [9,10] provide structural and behavioral details of graphs with maximum Laplacian spectral radius among all bipartite connected graphs of given order and size, and also determine those graphs which maximize the Laplacian spectral radius among all bipartite graphs with (edge-)connectivity at most k.

Let G be a simple graph. A *clique* of G is a subset of vertices such that it induces a complete subgraph of G. We denote the maximum clique size of G by ω which is called the *clique number* of G. Let $\mathscr{G}_{n,\omega}$ be the set of all connected graphs of order n with clique number ω, where $2 \leqslant \omega \leqslant n$. It is easy to see that $\mathscr{G}_{\omega,\omega} = \{K_\omega\}$ and if $G \in \mathscr{G}_{\omega+1,\omega}$, then G contains $K_{1,\omega}$ as a subgraph. Thus from Lemma 2.2, we have $\mu(G) = \omega + 1$. In the following, we consider that $n \geq \omega + 2$. The *kite graph* $PK_{n-\omega,\omega}$ (shown in Fig. 1) is a graph on n vertices obtained from the path $P_{n-\omega}$ and the complete graph K_ω by adding an edge between an end vertex of $P_{n-\omega}$ and a vertex of K_ω. Clearly, $PK_{n-2,2} = P_n$ and $PK_{0,n} = K_n$. In [19], it was shown that among all connected graphs with clique number ω, the minimum value of the spectral radius, which is the largest eigenvalue of the adjacency matrix of a graph, is attained from a kite graph $PK_{n-\omega,\omega}$. In [17], the first seven smallest Laplacian spectral radii among graphs in $\mathscr{G}_{n,\omega}$ can be obtained when $\omega = 2$ and $n \geq 17$. In this paper, the graphs with the first three smallest Laplacian spectral radii are determined among all graphs in $\mathscr{G}_{n,\omega}$, $(\omega \geq 3)$.

2. Preliminaries

Let G be a graph and let $G' = G + e$ be the graph obtained from G by inserting a new edge e into G. It follows by the well-known *Courant-Weyl* inequalities (see, e.g. [2, Theorem 2.1]) that the following is true.

Lemma 2.1. *The Laplacian eigenvalues of G and G' interlace, that is,*

$$\mu_1(G') \geqslant \mu_1(G) \geqslant \mu_2(G') \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0.$$

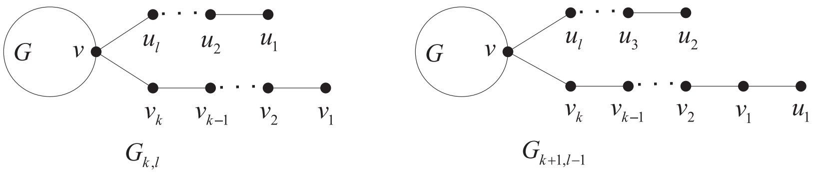

Fig. 2. Obtaining $G_{k+1,l-1}$ from $G_{k,l}$ by grafting an edge.

Lemma 2.2 [11,13]. *Let G be a connected graph on n vertices with at least one edge, then* $\mu(G) \geq \Delta(G)+1$, *where* $\Delta(G)$ *is the maximum degree of the graph G, with equality if and only if* $\Delta(G) = n-1$.

Lemma 2.3 [3]. *Let G be a graph on n vertices. Then*

$$\mu(G) \leq \max\{d(v_i) + d(v_j) - |N(v_i) \cap N(v_j)| : v_iv_j \in E\},$$

where $N(v)$ *denotes the neighborhood of the vertex v.*

An *internal path* of a graph G is a cycle $v_1v_2 \cdots v_{k-1}v_k$ with $v_1 = v_k$ or a path $v_1v_2 \cdots v_k$, where $k \geq 2$, such that $d(v_1) \geq 3$, $d(v_k) \geq 3$ and $d(v_i) = 2$ for $i \neq 1, k$.

For the Laplacian spectral radius, we have the following result.

Lemma 2.4 [6]. *Let* $P = v_1v_2 \cdots v_k$ *be an internal path of a connected bipartite graph G of order n, where* $n > k \geq 2$. *Let* G' *be a graph obtained from G by subdividing some edges of P. Then we have* $\mu(G') < \mu(G)$.

For $v \in V(G)$, let $L_v(G)$ be the principal sub-matrix of $L(G)$ obtained by deleting the row and column corresponding to the vertex v. The following lemma is often used to calculate the Laplacian characteristic polynomial of G.

Lemma 2.5 [4]. *Let* $G = G_1u : vG_2$ *be the graph obtained by joining the vertex u of* G_1 *and the vertex v of* G_2 *with an edge. Then*

$$\Phi(L(G)) = \Phi(L(G_1))\Phi(L(G_2)) - \Phi(L(G_1))\Phi(L_v(G_2)) - \Phi(L(G_2))\Phi(L_u(G_1)).$$

Let v be a vertex of a graph G and suppose that two new paths $P = vv_k \cdots v_2v_1$ and $Q = vu_l \cdots u_2u_1$ of length k and l $(k \geqslant l \geqslant 1)$ are attached to G at v, respectively, to form a new graph $G_{k,l}$ (shown in Fig. 2), where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_l$ are distinct. Let

$$G_{k+1,l-1} = G_{k,l} - u_1u_2 + v_1u_1.$$

We call that $G_{k+1,l-1}$ is obtained from $G_{k,l}$ by grafting an edge (see Fig. 2).

Lemma 2.6 [5]. *Let G be a connected graph on* $n \geq 2$ *vertices and v is a vertex of G. Let* $G_{k,l}$ *and* $G_{k+1,l-1}$ $(k \geqslant l \geqslant 1)$ *be the graphs as defined above. Then*

$$\mu(G_{k,l}) \geqslant \mu(G_{k+1,l-1}),$$

with equality if and only if there exists a unit eigenvector of $G_{k,l}$ *corresponding to* $\mu(G_{k,l})$ *taking the value* 0 *on vertex v.*

Lemma 2.7 [5,15]. *Let v be a vertex of a connected graph G and suppose that* $v_1, \ldots, v_s$ *are pendant vertices of G which are adjacent to v. Let* G^* *be the graph obtained from G by adding any t* $(1 \leq t \leq \frac{s(s-1)}{2})$ *edges among* $v_1, v_2 \ldots, v_s$. *Then we have* $\mu(G) = \mu(G^*)$.

Lemma 2.8. *Let u, v be two vertices of G and* $uv \notin E(G)$, *and* $\mathbf{x}$ *be a unit eigenvector of G corresponding to* $\mu(G)$. *If* $x_u \neq x_v$, *then* $\mu(G) < \mu(G + uv)$.

Proof. From Eq. (1.1), we have $\mu(G) = \boldsymbol{x}^T L(G)\boldsymbol{x} < \boldsymbol{x}^T L(G+uv)\boldsymbol{x} \leq \mu(G+uv)$. $\square$

3. The case $3 \leq \omega \leqslant n-5$

In this section, we consider the case $3 \leq \omega \leq n-5$.

Lemma 3.1. *Let $f_1(x)$ and $f_2(x)$ be two polynomials with positive leading coefficients and with real roots $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $r_1 \geq r_2 \geq \cdots \geq r_n$, respectively. Let $f_3(x)$ be another polynomial with real roots $s_1 \geq s_2 \geq \cdots \geq s_t$. Suppose $s_1 \geq \max\{\lambda_2, r_2\}$. If $k_1 f_1(s_1) + k_2 f_2(s_1) > 0$, where k_1 and k_2 are positive numbers, then either $s_1 > \lambda_1$ or $s_1 > r_1$.*

Proof. We prove the contrapositive of the statement. Suppose that $\lambda_1 \geq s_1$ and $r_1 \geq s_1$. Since the leading coefficients of $f_1(x)$ and $f_2(x)$ are positive and $s_1 \geq \max\{\lambda_2, r_2\}$, $f_1(s_1) \leq 0$ and $f_2(s_1) \leq 0$. Then we have $k_1 f_1(s_1) + k_2 f_2(s_1) \leq 0$. $\square$

Let H_1 be the graph obtained from K_ω and the path $P_4 = v_1v_2v_3v_4$ by joining a vertex of K_ω and a non-pendant vertex, say v_2, of P_4 by a path with length 2; $H_2 = H_1 + v_1v_3$ (see Fig. 3).

Lemma 3.2. *Let H_1 and H_2 be graphs defined above. Then $\mu(H_2) > \mu(H_1) > \omega + 1$.*

Proof. From Lemmas 2.1 and 2.2, we have $\mu(H_2) \geq \mu(H_1) > \omega + 1$. Let X be a unit eigenvector of H_1 corresponding to $\mu(H_1)$. From $(D-A)X = \mu(H_1)X$, we have

$$\begin{cases} (2-\mu(H_1))x_3 &= x_2 + x_4, \\ (1-\mu(H_1))x_4 &= x_3, \\ (1-\mu(H_1))x_1 &= x_2. \end{cases} \tag{3.1}$$

Then

$$[1 - 3\mu(H_1) + \mu(H_1)^2]x_3 = [1-\mu(H_1)]^2 x_1. \tag{3.2}$$

If $x_1 = x_3 = 0$, then from Eq. (3.1) we have $x_2 = x_4 = 0$. Thus $\mu(H_1)$ is a Laplacian eigenvalue of $H_1 - v_1 - v_2 - v_3 - v_4$. Then from Lemma 2.1, $\mu(H_1) = \mu(H_1 - v_1 - v_2 - v_3 - v_4) = \omega + 1$, a contradiction. If $x_1 = x_3 \neq 0$, then from Eq. (3.2), we have $\mu(H_1) = 0$, a contradiction. So, we have $x_1 \neq x_3$. From Lemma 2.8, we have $\mu(H_2) > \mu(H_1)$. $\square$

Let H_3 be the graph obtained from K_ω and P_5 by adding two edges between a pendant vertex of P_5 and two vertices of K_ω (see Fig. 3).

Lemma 3.3. *Let H_1, H_2 and H_3 be graphs defined in Fig. 3. Then $\omega + 2 > \mu(H_3) > \mu(H_1)$ for $\omega \geq 3$.*

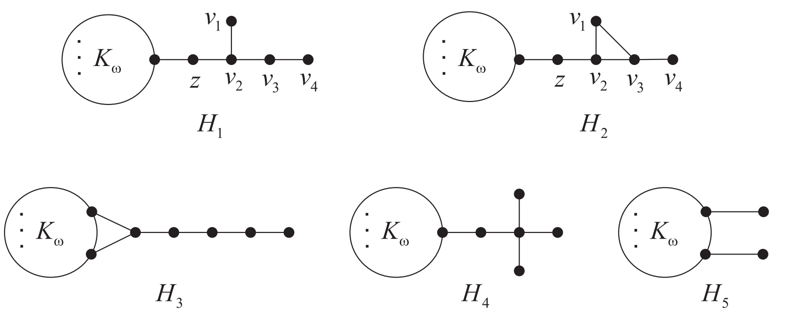

Fig. 3. Graphs H_1 to H_5.

Proof. From Lemma 2.5, by complicated and tedious computations, we have

$$\begin{aligned}\Phi(H_1) &= x(x-\omega)^{\omega-2}\big[x^6-(\omega+10)x^5+(9\omega+36)x^4\\ &\quad -(27\omega+60)x^3+(32\omega+51)x^2-(13\omega+24)x+\omega+5\big]\\ &= x(x-\omega)^{\omega-2}f_1(x).\\ \Phi(H_2) &= x(x-\omega)^{\omega-2}\big[x^6-(\omega+12)x^5+(11\omega+53)x^4\\ &\quad -(42\omega+108)x^3+(65\omega+109)x^2-(36\omega+58)x+3\omega+15\big]\\ &= x(x-\omega)^{\omega-2}f_2(x).\\ \Phi(H_3) &= x(x-\omega-1)(x-\omega)^{\omega-3}\big[x^6-(\omega+11)x^5+(10\omega+45)x^4\\ &\quad -(35\omega+87)x^3+(50\omega+87)x^2-(25\omega+47)x+2\omega+10\big]\\ &= x(x-\omega-1)(x-\omega)^{\omega-3}f_3(x).\end{aligned}$$

From Lemma 2.2, it is easy to see that $\mu(H_i) > \omega+1$ $(i=1,2,3)$. Thus $\mu(H_1)$, $\mu(H_2)$ and $\mu(H_3)$ are the largest roots of the equations $f_1(x)=0$, $f_2(x)=0$ and $f_3(x)=0$, respectively. Note that $f_2(\omega+2)\neq 0$ and $f_3(\omega+2)\neq 0$. From Lemma 2.3, we have $\mu(H_2)<\omega+2$ and $\mu(H_3)<\omega+2$. Then for $\omega+1<x<\omega+2$ we have,

$$\begin{aligned}f_1(x)+f_2(x)-2f_3(x) &= -x^4+(\omega+6)x^3-(3\omega+14)x^2+(\omega+12)x\\ &= x[-x^3+(\omega+6)x^2-(3\omega+14)x+\omega+12]\\ &> x^2[-x^2+(\omega+6)x-(3\omega+13)]\\ &> x^2[-x^2+(\omega+2)x+\omega-9]\\ &> 0 \quad (\text{when } \omega\geq 9).\end{aligned}$$

So $f_1(\mu(H_3))+f_2(\mu(H_3)) = f_1(\mu(H_3))+f_2(\mu(H_3))-2f_3(\mu(H_3)) > 0$ for $\omega\geq 9$. It is easy to calculate that $\Phi(K_\omega)=x(x-\omega)^{\omega-1}$. Then from Lemmas 2.1 and 2.2, we have $\mu_2(H_1)\leq\mu(H_1-v_2z)=\omega+1<\mu(H_3)$ and $\mu_2(H_2)\leq\mu(H_2-v_2z)=\omega+1<\mu(H_3)$, where z is the vertex indicated in Fig. 3. Thus from Lemmas 3.1 and 3.2, we have for $\omega\geq 9$, $\mu(H_3)>\mu(H_1)$. When $3\leq\omega\leq 8$. It is easy to show that $\mu(H_3)>\mu(H_1)$ by direct calculation. □

Let H_4 be the graph obtained from K_ω and a star $K_{1,4}$ by adding an edge between a vertex of K_ω and a pendant vertex of $K_{1,4}$; and H_5 be the graph obtained from K_ω and two isolated vertices, say u_1, u_2 by adding two edges v_1u_1 and v_2u_2, where v_1, v_2 are two different vertices of K_ω (see Fig. 3).

Lemma 3.4. *Let H_4 and H_5 be graphs defined in Fig. 3. Then $\mu(H_5)>\mu(H_4)$ for $\omega\geq 5$.*

Proof. From Lemma 2.5, by complicated and tedious computations, we have

$$\begin{aligned}\Phi(H_4) &= x(x-1)^2(x-\omega)^{\omega-2}[x^4-(\omega+8)x^3+(7\omega+17)x^2-(10\omega+12)x+\omega+5]\\ &= x(x-1)^2(x-\omega)^{\omega-2}g_1(x).\\ \Phi(H_5) &= x(x-\omega)^{\omega-3}[x^4-(2\omega+4)x^3+(\omega^2+6\omega+6)x^2-(2\omega^2+6\omega+4)x+\omega^2+2\omega]\\ &= x(x-\omega)^{\omega-3}g_2(x).\end{aligned} \tag{3.3}$$

From Lemma 2.2, we have $\mu(H_4)>\omega+1$ and $\mu(H_5)>\omega+1$. Thus $\mu(H_4)$ and $\mu(H_5)$ are the largest roots of the equations $g_1(x)=0$ and $g_2(x)=0$, respectively. From Lemma 2.1, we have $\mu_2(H_4)\leq\omega+1$ and $\mu_2(H_5)\leq\omega+1$. By direct calculation, we have

$$g_1(\omega+1)=3-\omega<0,\quad (\omega\geq 5),$$

$$g_1(\omega+1+\frac{1}{2\omega}) = -3\omega+\frac{1}{2}\omega^2-\frac{1}{\omega}-\frac{1}{2\omega^3}+\frac{1}{8\omega^2}+\frac{1}{16\omega^4}+\frac{13}{4}$$
$$> -3\omega+\frac{1}{2}\omega^2-\frac{1}{\omega}+\frac{13}{4} > -3\omega+\frac{1}{2}\omega^2+3>0, \qquad (\omega\geq 5).$$

Then $\mu(H_4) < \omega+1+\frac{1}{2\omega}$.

Similarly, we have

$$g_2(\omega+1+\frac{1}{2\omega}) = -\frac{3}{4}+\frac{1}{4\omega^2}+\frac{1}{16\omega^4}<0.$$

$$g_2(\omega+2)=\omega^2+2\omega>0.$$

Then we have for $\omega \geq 5$, $\mu(H_5) > \omega+1+\frac{1}{2\omega} > \mu(H_4)$. □

Corollary 3.5. *Let H_1 and H_5 be graphs defined in Fig. 3. Then $\mu(H_5) > \mu(H_1)$ for $\omega \geq 4$.*

Proof. From Lemmas 2.6 and 3.4, we have $\mu(H_5) > \mu(H_4) \geq \mu(H_1)$ when $\omega \geq 5$. When $\omega = 4$, by direct calculation, we have $\mu(H_5) > \mu(H_1)$. □

Let $PK^i_{n-\omega,\omega}$ be the graph obtained from the kite graph $PK_{n-\omega-1,\omega}$ (see Fig. 1) and an isolated vertex v_n by adding an edge v_nv_i $(\omega+1 \leq i \leq n-1)$ (see Fig. 4). It is easy to see that $PK^{\omega+2}_{5,\omega} = H_1$ and $PK^{n-1}_{n-\omega,\omega} = PK_{n-\omega,\omega}$.

Let $\overline{PK}^{n-2}_{n-\omega,\omega} = PK^{n-2}_{n-\omega,\omega} + v_{n-1}v_n$ and $\overline{PK}^{n-3}_{n-\omega,\omega} = PK^{n-3}_{n-\omega,\omega} + v_{n-2}v_n$ (see Fig. 5).

Lemma 3.6. *Let $PK_{n-\omega,\omega}$, $PK^{n-2}_{n-\omega,\omega}$, $\overline{PK}^{n-2}_{n-\omega,\omega}$, $PK^{n-3}_{n-\omega,\omega}$ and $\overline{PK}^{n-3}_{n-\omega,\omega}$ be graphs defined in Figs. 1, 4 and 5. Then*

$$\mu(\overline{PK}^{n-3}_{n-\omega,\omega}) > \mu(PK^{n-3}_{n-\omega,\omega}) > \mu(\overline{PK}^{n-2}_{n-\omega,\omega}) = \mu(PK^{n-2}_{n-\omega,\omega}) > \mu(PK_{n-\omega,\omega}).$$

Proof. By a similar proof of Lemma 3.2, we have $\mu(\overline{PK}^{n-3}_{n-\omega,\omega}) > \mu(PK^{n-3}_{n-\omega,\omega})$. Let $PS^{n-2}_{n-\omega,\omega}$ and $PS^{n-3}_{n-\omega,\omega}$ be graphs obtained from $PK^{n-2}_{n-\omega,\omega}$ and $PK^{n-3}_{n-\omega,\omega}$ by deleting all the edges among $v_2, v_3, \ldots, v_\omega$, respectively. Then both $PS^{n-2}_{n-\omega,\omega}$ and $PS^{n-3}_{n-\omega,\omega}$ are trees. From Lemmas 2.1, 2.4 and 2.7, we have

$$\mu(PK^{n-3}_{n-\omega,\omega}) = \mu(PS^{n-3}_{n-\omega,\omega}) > \mu(PS^{n-2}_{n+1-\omega,\omega})$$
$$\geq \mu(PS^{n-2}_{n-\omega,\omega}\cup K_1) = \mu(PS^{n-2}_{n-\omega,\omega}) = \mu(PK^{n-2}_{n-\omega,\omega}). \tag{3.4}$$

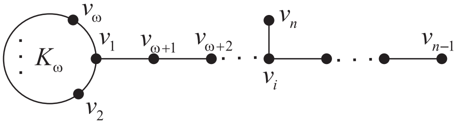

Fig. 4. Graph $PK^i_{n-\omega,\omega}$, where $i = \omega+1, \ldots, n-1$.

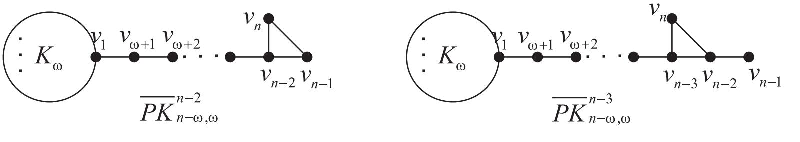

Fig. 5. Graphs $\overline{PK}^{n-2}_{n-\omega,\omega}$ and $\overline{PK}^{n-3}_{n-\omega,\omega}$.

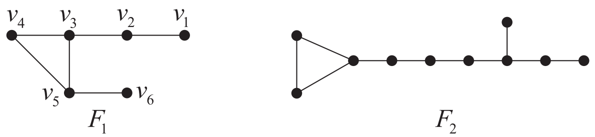

Fig. 6. Graphs F_1 and F_2.

From Lemma 2.7, we see that $\mu(PK^{n-2}_{n-\omega,\omega}) = \mu(\overline{PK}^{n-2}_{n-\omega,\omega})$. From Lemma 2.6, we have $\mu(PK^{n-2}_{n-\omega,\omega}) > \mu(PK_{n-\omega,\omega})$. The proof is completed. $\square$

Now we give the main result of the paper.

Theorem 3.7. *Among all connected graphs on $n \geq 10$ vertices with clique number ω, where $3 \leq \omega \leq n-5$, the first three smallest Laplacian spectral radii are obtained from $PK_{n-\omega,\omega}$; $PK^{n-2}_{n-\omega,\omega}$, $\overline{PK}^{n-2}_{n-\omega,\omega}$; and $PK^{n-3}_{n-\omega,\omega}$, respectively.*

Proof. Let G be a connected graph with clique number $\omega \geq 3$ and $n \geq \omega + 5$ vertices. Suppose that K_ω is a maximum clique of G. From Lemma 3.6, we only need to prove that if $G \notin \{PK_{n-\omega,\omega}, PK^{n-2}_{n-\omega,\omega}, \overline{PK}^{n-2}_{n-\omega,\omega}, PK^{n-3}_{n-\omega,\omega}\}$, then $\mu(G) > \mu(PK^{n-3}_{n-\omega,\omega})$. We separate into the following three cases:
Case 1. If there exist at least two vertices outside of K_ω that are adjacent some vertices of K_ω, then either $\Delta(G) \geq \omega + 1$ or G contains H_5 as a subgraph. If $\Delta(G) \geq \omega + 1$, then from Lemmas 2.2 and 3.3, we have $\mu(G) \geq \omega + 2 > \mu(H_1)$. By a similar argument as that of Eq. (3.4), we have

$$\mu(H_1) \geq \mu(PS^{n-3}_{n-\omega,\omega}) = \mu(PK^{n-3}_{n-\omega,\omega}). \tag{3.5}$$

Thus we have $\mu(G) > \mu(PK^{n-3}_{n-\omega,\omega})$. Suppose G contains H_5 as a subgraph and $\Delta(G) = \omega$. If $\omega \geq 4$, then from Lemma 2.1, Corollary 3.5 and Eq. (3.5), we have,

$$\mu(G) \geq \mu(H_5) > \mu(H_1) \geq \mu(PK^{n-3}_{n-\omega,\omega}).$$

If $\omega = 3$, then $n \geq 8$ and G contains F_1 as a subgraph, where F_1 is the graph obtained from the path $P_6 = v_1v_2v_3v_4v_5v_6$ by adding the edge v_3v_5 (see Fig. 6). By direct calculation, it is easy to see that $\mu(F_1) \approx 4.39276 > 4.37213 \approx \mu(F_2)$, where $F_2 = PK^7_{7,3}$ (see Fig. 6). By a similar argument as that of Eq. (3.4), we have $\mu(F_2) \geq \mu(PK^{n-3}_{n-3,3})$ for $n \geq 10$. Thus from Lemma 2.1, we have

$$\mu(G) \geq \mu(F_1) > \mu(F_2) \geq \mu(PK^{n-3}_{n-3,3}),$$

for $n \geq 10$.
Case 2. Suppose that there exists a vertex u outside of K_ω such that u is adjacent to at least two vertices of K_ω and Case 1 does not occur. Let G_1 be the graph obtained from K_ω and an isolated vertex u by adding two edges between u and two different vertices of K_ω together with a tree T with $n-\omega$ vertices attached to u (see Fig. 7). Then G_1 is a spanning subgraph of G. By Lemma 2.1, we have $\mu(G) \geq \mu(G_1)$. From Lemma 2.6, we have $\mu(G_1) \geq \mu(H_3')$, where H_3' is the graph obtained from K_ω and the path $P_{n-\omega}$ by adding two edges between an end vertex of $P_{n-\omega}$ and two different vertices of K_ω. Note that, it is easy to see that $H_3' = H_3$ when $n - \omega = 5$ (see Figs. 3 and 7). From Lemma 2.1, we have $\mu(H_3') \geq \mu(H_3)$. Then we have $\mu(G) \geq \mu(H_3)$. From Lemma 3.3 and Eq. (3.5), we have

$$\mu(G) \geq \mu(H_3) > \mu(H_1) \geq \mu(PK^{n-3}_{n-\omega,\omega}).$$

Case 3. There is only one vertex u outside of K_ω that is adjacent to exactly one vertex of K_ω. Suppose that $G - V(K_\omega)$ is a tree. Since $G \notin \{PK_{n-\omega,\omega}, PK^{n-2}_{n-\omega,\omega}, PK^{n-3}_{n-\omega,\omega}\}$, from Lemma 2.6 and a similar argument as that of Eq. (3.4), we have $\mu(G) > \mu(PK^{n-3}_{n-\omega,\omega})$.

Suppose that C_g ($g \geq 3$) is one of the largest cycles of $G - V(K_\omega)$. Let G_g^k be the graph obtained from K_ω and the cycle C_g by joining a vertex of K_ω and a vertex of C_g with a path with length $k+1$,

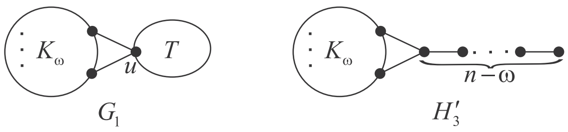

Fig. 7. Graphs G_1 and H_3'.

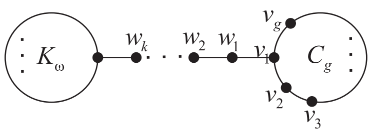

Fig. 8. Graph G_g^k.

where $0 \le k \le n-g-\omega$. Since $G_g = G_g^k \cup N_{n-g-\omega-k}$ (here N_h denotes the null graph of order h) is a spanning subgraph of G for some suitable k, by Lemma 2.1 we have $\mu(G) \ge \mu(G_g) = \mu(G_g^k)$. Suppose that $C_g = v_1v_2\cdots v_gv_1$ and $d_{G_g^k}(v_1) = 3$ (see Fig. 8). We consider the following three subcases:

Subcase 3.1. Suppose that $g \ge 5$. From Lemma 2.1, we have $\mu(G_g) \ge \mu(G_g - v_2v_3)$. By a similar argument as that of Eq. (3.4), we have $\mu(G_g - v_2v_3) > \mu(PK_{n-\omega,\omega}^{n-3})$. Thus we have $\mu(G) > \mu(PK_{n-\omega,\omega}^{n-3})$.

Subcase 3.2. Suppose that $g = 4$. Consider the graph $G_4 - v_2v_3$. Let X be a unit eigenvector of $G_4 - v_2v_3$ corresponding to $\mu = \mu(G_4 - v_2v_3)$. Note that the coordinates x_i of X is corresponding to v_i, for $1 \le i \le 4$. From $(D-A)X = \mu X$, we have

$$\begin{cases} (2-\mu)x_4 = x_1 + x_3, \\ (1-\mu)x_2 = x_1, \\ (1-\mu)x_3 = x_4. \end{cases}$$

Then we have

$$[(2-\mu)(1-\mu)-1]x_3 = (1-\mu)x_2. \tag{3.6}$$

If $x_2 = x_3 = 0$, then we have from $(D-A)X = \mu X$ that $x_1 = x_4 = x_z = 0$, where z is any vertex of the path joining K_ω and C_4 of G_4. Thus we have $\mu = \mu(G_4 - v_2v_3) = \mu(K_\omega) = \omega$, a contradiction.

If $x_2 = x_3 \ne 0$, then from Eq. (3.6), we have $\mu = 0$ or $\mu = 2$, a contradiction. So, we have $x_2 \ne x_3$. From Lemma 2.8, we have $\mu(G_4) > \mu(G_4 - v_2v_3)$. By a similar argument as that of Eq. (3.4), we have $\mu(G_4 - v_2v_3) \ge \mu(PK_{n-\omega,\omega}^{n-3})$. Thus we have $\mu(G) > \mu(PK_{n-\omega,\omega}^{n-3})$.

Subcase 3.3. Suppose that $g = 3$. Since $G \ne \overline{PK}_{n-\omega,\omega}^{n-2}$, G contains a spanning subgraph $G_3 = G_3^k \cup N_{n-3-\omega-k}$ for some k with $0 \le k \le n-\omega-4$. Suppose that the path joining K_ω and C_3 of G_3 is $v_1w_1w_2\cdots w_ku$, where v_1 is a vertex of C_3 and u is a vertex of K_ω. Since G is connected and $N_{n-3-\omega-k}$

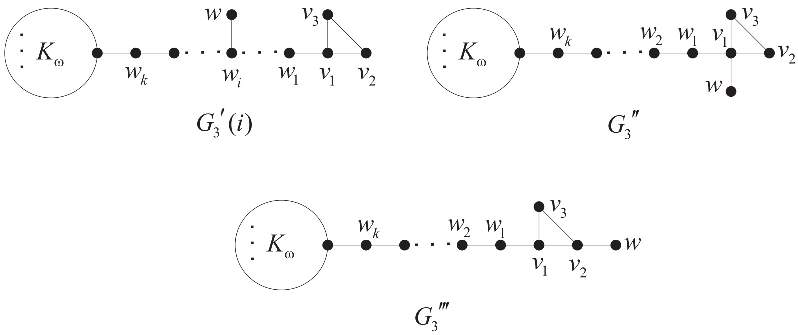

Fig. 9. Graphs $G_3'(i)$ $(1 \le i \le k)$, G_3'' and G_3'''.

contains at least one vertex, say w, G must contain one of the graphs $G_3'(i)$ $(1 \le i \le k)$, G_3'' and G_3''' as a spanning subgraph, where $G_3'(i) = G_3 + w_i w$, $G_3'' = G_3 + v_1 w$ and $G_3''' = G_3 + v_2 w$ (see Fig. 9).

If G contains $G_3'(i)$, then from Lemmas 2.1 and 2.7, we have $\mu(G) \ge \mu(G_3'(i)) = \mu(G_3'(i) - v_2v_3)$. From Lemma 2.6, we have $\mu(G_3'(i) - v_2v_3) > \mu(G_3'(i) - v_2v_3 - v_1v_3 + v_2v_3)$. By a similar argument as that of Eq. (3.4), we have $\mu(G_3'(i) - v_2v_3 - v_1v_3 + v_2v_3) > \mu(PK_{n-\omega,\omega}^{n-3})$. Thus we have $\mu(G) > \mu(PK_{n-\omega,\omega}^{n-3})$.

If G contains G_3'', then from Lemmas 2.1 and 2.7, we have $\mu(G) \ge \mu(G_3'') = \mu(G_3'' + wv_2)$. Note that $G_3'' + wv_2 - V(K_\omega)$ contains C_4 as a subgraph. By a similar argument as that of Subcase 3.2, we have $\mu(G_3'' + wv_2) > \mu(PK_{n-\omega,\omega}^{n-3})$. Thus we have $\mu(G) > \mu(PK_{n-\omega,\omega}^{n-3})$.

If G contains G_3''', then consider the graph $G_3''' - v_2v_3$. By a similar argument as the proof of Lemma 3.2, we have $\mu(G_3''') > \mu(G_3''' - v_2v_3)$. From Lemma 2.1, we have $\mu(G) \ge \mu(G_3''') > \mu(G_3''' - v_2v_3)$. By a similar argument as that of Eq. (3.4), we have $\mu(G_3''' - v_2v_3) \ge \mu(PK_{n-\omega,\omega}^{n-3})$. Thus we have $\mu(G) > \mu(PK_{n-\omega,\omega}^{n-3})$. $\square$

4. The case $\omega = n-2$, $n-3$ and $n-4$

In what follows, we consider the cases $n = \omega + 2$, $n = \omega + 3$ and $n = \omega + 4$, respectively.

Let M_ω^t be the graph obtained from K_ω and an edge uv by adding t edges between u and t vertices of K_ω $(\omega \ge t)$ (see Fig. 10). It is easy to see that $M_\omega^1 = PK_{2,\omega}$.

Lemma 4.1. *Let H_5 be the graph defined in Fig. 3. Then we have*

$$\mu(H_5) = \mu(M_\omega^t) \text{ for } \omega = 2t;$$
$$\mu(H_5) > \mu(M_\omega^t) \text{ for } \omega > 2t.$$

Moreover, $\mu(M_\omega^t) > \mu(M_\omega^{t-1})$.

Proof. By direct calculation, we have

$$\begin{aligned}\Phi(M_\omega^t) &= x(x-\omega)^{\omega-t-1}(x-\omega-1)^{t-1}[x^3-(\omega+t+3)x^2 \\ &\quad + (t\omega+2\omega+2t+2)x - t\omega - 2t] \\ &= x(x-\omega)^{\omega-t-1}(x-\omega-1)^{t-1}h_t(x).\end{aligned}$$

By a similar proof of Lemma 3.4, we have $\mu_2(M_\omega^t) \le \omega + 1 < \mu(M_\omega^t)$ and $\mu(M_\omega^t)$ is the largest root of the equation $h_t(x) = 0$. If $\omega = 2t$, then

$$h_t(x) = x^3 - (3t+3)x^2 + (2t^2+6t+2)x - 2t^2 - 2t = (x-t-1)[x^2-(2t+2)x+2t].$$

From the proof of Lemma 3.4, we have known that $\mu(H_5)$ is the largest root of the equation $g_2(x) = x^4 - (2\omega+4)x^3 + (\omega^2+6\omega+6)x^2 - (2\omega^2+6\omega+4)x + \omega^2 + 2\omega = 0$. If $\omega = 2t$, then

$$\begin{aligned}g_2(x) &= x^4 - (4t+4)x^3 + (4t^2+12t+6)x^2 - (8t^2+12t+4)x + 4t^2 + 4t \\ &= [x^2-(2t+2)x+2t][x^2-(2t+2)x+2t+2].\end{aligned}$$

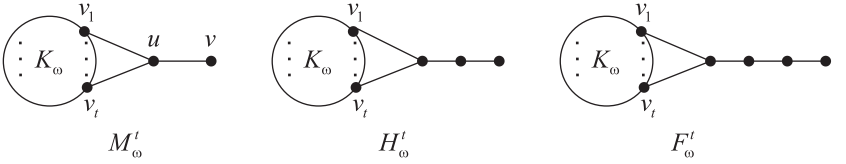

Fig. 10. Graphs M_ω^t, H_ω^t and F_ω^t.

It is easy to check that $h_t(x) = 0$ and $g_2(x) = 0$ have the same largest root when $\omega = 2t$. Thus we have $\mu(H_5) = \mu(M^t_\omega)$ for $\omega = 2t$.

If $\omega > 2t$, then

$$h_t(\omega+1) = -t < 0;$$
$$h_t(\omega+1+\frac{1}{\omega+1}) = \omega - 2t - 1 + \frac{t+2}{\omega+1} - \frac{t+2}{(\omega+1)^2} + \frac{1}{(\omega+1)^3} > 0$$

and

$$g_2(\omega+1+\frac{1}{\omega+1}) = \frac{1}{\omega+1}(\frac{3}{\omega+1} + \frac{1}{(\omega+1)^3} - 2 - \frac{2}{(\omega+1)^2}) < 0;$$
$$g_2(\omega+2) = \omega^2 + 2\omega > 0.$$

Thus, we have $\mu(H_5) > \omega + 1 + \frac{1}{\omega+1} > \mu(M^t_\omega)$ for $\omega > 2t$.

Since $h_{t-1}(x) - h_t(x) = (x-\omega-1)(x-1)+1 > 0$ when $x > \omega+1$, we have $\mu(M^t_\omega) > \mu(M^{t-1}_\omega)$. □

Theorem 4.2. *Among all connected graphs on $n \geq 9$ vertices with clique number $\omega = n-2$, the first three smallest Laplacian spectral radii are obtained from M^1_ω $(= PK_{2,\omega})$, M^2_ω and M^3_ω, respectively.*

Proof. Let K_ω be the clique of G. From the proof of Case 1 of Theorem 3.7 and Lemma 4.1, we have to only deal with the case that there exists exactly one vertex u outside of K_ω such that u is adjacent to some vertices of K_ω. That is, $G = M^t_\omega$ for some t. The result follows from Lemma 4.1. □

Let H^t_ω be the graph obtained from K_ω and P_3 by adding t edges between a pendant vertex of P_3 and t vertices of K_ω $(\omega \geq t)$ (see Fig. 10). It is easy to see that $H^1_\omega = PK_{3,\omega}$.

Lemma 4.3. *Let H_5 be the graph defined in Fig. 3. Then we have*

$$\mu(H_5) > \mu(H^t_\omega) \text{ for } \omega > 2t \text{ and } \mu(H^t_\omega) > \mu(H^{t-1}_\omega).$$

Proof. By direct calculation, we have

$$\begin{aligned}\Phi(H^t_\omega) &= x(x-\omega)^{\omega-t-1}(x-\omega-1)^{t-1}[x^4 - (\omega+t+5)x^3 + (t\omega+4\omega+4t+7)x^2 \\ &\quad - (3t\omega+5t+3\omega+3)x + t\omega+3t] \\ &= x(x-\omega)^{\omega-t-1}(x-\omega-1)^{t-1}p_t(x).\end{aligned}$$

By a similar proof of Lemma 3.4, we have $\mu_2(H^t_\omega) \leq \omega+1 < \mu(H^t_\omega)$ and $\mu(H^t_\omega)$ is the largest root of the equation $p_t(x) = 0$. If $\omega > 2t$, then

$$p_t(\omega+1) = t - t\omega < 0$$
$$\begin{aligned}p_t(\omega+1+\frac{1}{\omega+1}) &= \omega^2 - 2t\omega - 2\omega + 3t + 3 \\ &\quad - \frac{4t+8}{\omega+1} + \frac{3t+6}{(\omega+1)^2} - \frac{t+4}{(\omega+1)^3} + \frac{1}{(\omega+1)^4} \\ &> \omega^2 - 2t\omega - 2\omega + 3t \\ &= (\omega-1)(\omega-2t-1) + t - 1 \geq 0.\end{aligned}$$

Thus we have $\mu(H^t_\omega) < \omega + 1 + \frac{1}{\omega+1}$. From the proof of Lemma 4.1, we have known that for $\omega > 2t$, $\mu(H_5) > \omega + 1 + \frac{1}{\omega+1}$. So for $\omega > 2t$, we have $\mu(H_5) > \mu(H^t_\omega)$.

Since $p_{t-1}(x) - p_t(x) = x(x-3)(x-\omega-1) + 2x - \omega - 3 > 0$ when $x > \omega+1$, we have $\mu(H^t_\omega) > \mu(H^{t-1}_\omega)$. □

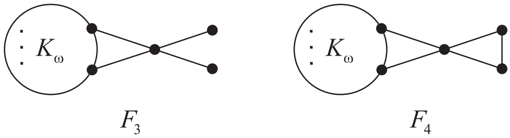

Fig. 11. Graphs F_3 and F_4.

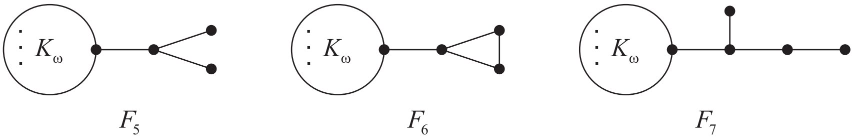

Fig. 12. Graphs F_5, F_6 and F_7.

Let F_3 be the graph obtained from K_ω and P_3 by adding two edges between the middle vertex of P_3 and two vertices of K_ω, and F_4 be the graph obtained from F_3 by adding an edge between two pendant vertices (see Fig. 11).

Lemma 4.4. *Let H_ω^t be the graph defined in Fig. 10. Then we have $\mu(F_3) = \mu(F_4) > \mu(H_\omega^3)$.*

Proof. It is easy to see that $\mu(F_3) = \mu(F_4)$ from Lemma 2.7. By direct calculation, we have

$$\begin{aligned}\Phi(F_3) &= x(x-1)(x-\omega)^{\omega-3}(x-\omega-1)[x^3-(\omega+6)x^2+(5\omega+7)x-2\omega-6]\\ &= x(x-1)(x-\omega)^{\omega-3}(x-\omega-1)g_3(x).\end{aligned}$$

From Lemma 2.2, we have $\mu(F_3) > \omega+1$. Then $\mu(F_3)$ is the largest root of the equation $g_3(x) = 0$.

From Lemma 4.3, we have $\mu(H_\omega^3)$ is the largest root of the equation

$$p_3(x) = x^4-(\omega+8)x^3+(7\omega+19)x^2-(12\omega+18)x+3\omega+9 = 0.$$

Since $p_3(x)-(x-2)g_3(x) = 2x-\omega-3 > 0$ when $x > \omega+1$, we have $\mu(F_3) > \mu(H_\omega^3)$. $\square$

Let F_5 be the graph obtained from K_ω and P_3 by adding an edge between the middle vertex of P_3 and a vertex of K_ω, and F_6 be the graph obtained from F_5 by adding an edge between two pendant vertices (see Fig. 12).

Lemma 4.5. *Let H_ω^t be the graph defined in Fig. 10. Then we have $\mu(F_5) = \mu(F_6) = \mu(H_\omega^2)$.*

Proof. It is easy to see that $\mu(F_5) = \mu(F_6)$ from Lemma 2.7. By direct calculation, we have

$$\Phi(F_5) = x(x-1)(x-\omega)^{\omega-2}[x^3-(\omega+5)x^2+(4\omega+5)x-\omega-3].$$

From Lemma 2.2, we have $\mu(F_5) > \omega+1$. Then $\mu(F_5)$ is the largest root of the equation

$$x^3-(\omega+5)x^2+(4\omega+5)x-\omega-3 = 0.$$

From Lemma 4.3, we have $\mu(H_\omega^2)$ is the largest root of the equation

$$\begin{aligned}p_2(x) &= x^4-(\omega+7)x^3+(6\omega+15)x^2-(9\omega+13)x+2\omega+6\\ &= (x-2)[x^3-(\omega+5)x^2+(4\omega+5)x-\omega-3] = 0.\end{aligned}$$

Thus, we have $\mu(F_5) = \mu(H_\omega^2)$. $\square$

Theorem 4.6. *Among all connected graphs on $n \geq 10$ vertices with clique number $\omega = n-3$, the first three smallest Laplacian spectral radii are obtained from H_ω^1 $(= PK_{3,\omega})$; H_ω^2, F_5, F_6; and H_ω^3, respectively.*

Proof. Let K_ω be the clique of G. From Lemma 4.3 and the proof of Case 1 of Theorem 3.7, we may assume that there exists exactly one vertex u outside of K_ω such that u is adjacent to some vertices of K_ω. We deal with the following three cases:

Case 1. If u is adjacent to t $(t \geq 3)$ vertices of K_ω, then from Lemmas 2.1 and 2.6, we have $\mu(G) \geq \mu(H_\omega^t)$. Moreover, the equality holds if and only if $G = H_\omega^t$. From Lemma 4.3, we have $\mu(H_\omega^t) > \mu(H_\omega^3)$ when $t \geq 4$. Thus we have $\mu(G) \geq \mu(H_\omega^3)$, the equality holds if and only if $G = H_\omega^3$.

Case 2. If u is adjacent to exactly two vertices of K_ω, then $G \in \{H_\omega^2, F_3, F_4\}$. From Lemmas 4.3 and 4.4, we have $\mu(F_3) = \mu(F_4) > \mu(H_\omega^3) > \mu(H_\omega^2)$.

Case 3. If u is adjacent to exactly one vertex of K_ω, then $G \in \{H_\omega^1, F_5, F_6\}$. From Lemmas 2.6 and 2.7, we have $\mu(F_5) = \mu(F_6) > \mu(H_\omega^1)$.

From Lemmas 4.3 and 4.5, we have $\mu(H_\omega^3) > \mu(F_5) = \mu(F_6) = \mu(H_\omega^2) > \mu(H_\omega^1)$. The result follows. $\square$

Let F_ω^t be the graph obtained from K_ω and P_4 by adding t edges between an end vertex of P_4 and t vertices of K_ω with $t \leq \omega$ (see Fig. 10). It is easy to see that $F_\omega^1 = PK_{4,\omega}$.

Lemma 4.7. *Let H_5 be the graph defined in Fig. 3. Then we have*

$$\mu(H_5) > \mu(F_\omega^t) \text{ for } \omega > 2t, \text{ and } \mu(F_\omega^t) > \mu(F_\omega^{t-1}).$$

Proof. By direct calculation, we have

$$\begin{aligned}\Phi(F_\omega^t) &= x(x-\omega)^{\omega-t-1}(x-\omega-1)^{t-1}[x^5-(\omega+t+7)x^4+(t\omega+6\omega+6t+16)x^3\\ &\quad -(5t\omega+12t+10\omega+14)x^2+(6t\omega+4\omega+11t+4)x-t\omega-4t]\\ &= x(x-\omega)^{\omega-t-1}(x-\omega-1)^{t-1}q_t(x).\end{aligned}$$

By a similar proof of Lemma 3.4, we have $\mu_2(F_\omega^t) \leq \omega+1 < \mu(F_\omega^t)$ and $\mu(F_\omega^t)$ is the largest root of the equation $q_t(x) = 0$. If $\omega > 2t$, then

$$\begin{aligned}q_t(\omega+1) &= 2t\omega - t\omega^2 < 0\\ q_t\left(\omega+1+\frac{1}{\omega+1}\right) &= \omega^3-2t\omega^2+5t\omega+5\omega-3\omega^2-7t-13\\ &\quad +\frac{15t+26}{\omega+1}-\frac{10t+22}{(\omega+1)^2}+\frac{5t+14}{(\omega+1)^3}-\frac{t+6}{(\omega+1)^4}+\frac{1}{(\omega+1)^5}\\ &> \omega^3-2t\omega^2+5t\omega+5\omega-3\omega^2-7t-13+\frac{15t+18}{\omega+1}\\ &= (\omega-2t-1)(\omega^2-2\omega+3)+\frac{t\omega^2-10\omega+14t+8}{\omega+1}\\ &\geq 0 \quad \text{when } t \geq 3.\end{aligned}$$

When $t = 2$. Then $\omega \geq 5$. The fraction $\frac{t\omega^2-10\omega+14t+8}{\omega+1} = \frac{2\omega^2-10\omega+36}{\omega+1} > 0$. When $t = 1$. Then $\omega \geq 3$. The fraction $\frac{t\omega^2-10\omega+14t+8}{\omega+1} = \frac{\omega^2-10\omega+22}{\omega+1} = \frac{(\omega-3)(\omega-7)+1}{\omega+1} > 0$ for $\omega \geq 7$. It is easy to check that $q_1(\omega+1+\frac{1}{\omega+1}) > 0$ for $3 \leq \omega \leq 6$.

Thus we have $\mu(F_\omega^t) < \omega+1+\frac{1}{\omega+1}$. From the proof of Lemma 4.1, we have known that for $\omega > 2t$, $\mu(H_5) > \omega+1+\frac{1}{\omega+1}$. So we have for $\omega > 2t$, $\mu(H_5) > \mu(F_\omega^t)$.

For $\omega+2 > x > \omega+1$ and $\omega \geq 4$, we have

$$\begin{aligned}q_{t-1}(x)-q_t(x) &= x^4-(\omega+6)x^3+(5\omega+12)x^2-(6\omega+11)x+\omega+4\\ &> x[x^3-(\omega+6)x^2+(5\omega+12)x-(6\omega+10)]\end{aligned}$$

$$> x^2[x^2 - (\omega + 6)x + (5\omega + 5)] + \omega - 3$$
$$= x^2(x - \omega - 1)(x - 5) + \omega - 3 \geq 0.$$

Hence $\mu(H_\omega^t) > \mu(H_\omega^{t-1}) \geq 4$. For $\omega = 3$, the result is easily to verify. $\square$

Let F_7 be the graph obtained from K_ω and P_4 by adding an edge between a vertex of P_4 of degree 2 and a vertex of K_ω (see Fig. 12).

Lemma 4.8. *Let F_ω^2 be the graph defined in Fig. 10. Then we have $\mu(F_7) > \mu(F_\omega^2)$.*

Proof. By direct calculation, we have

$$\begin{aligned}\Phi(F_7) &= x(x-\omega)^{\omega-2}[x^5 - (\omega+8)x^4 + (7\omega+21)x^3 \\ &\quad - (14\omega+24)x^2 + (8\omega+15)x - \omega - 4] \\ &= x(x-\omega)^{\omega-2}g_4(x).\end{aligned}$$

From Lemma 2.2, we have $\mu(F_7) > \omega + 1$. Then $\mu(F_7)$ is the largest root of the equation $g_4(x) = 0$. From the proof of Lemma 4.3, we have $\mu(F_\omega^2)$ is the largest root of the equation

$$q_2(x) = x^5 - (\omega+9)x^4 + (8\omega+28)x^3 - (20\omega+38)x^2 + (16\omega+26)x - 2\omega - 8 = 0.$$

Since

$$(x-1)q_2(x) - (x-2)g_4(x) = x(x-\omega) > 0, \ \text{when } x > \omega + 1,$$

we have $\mu(F_7) > \mu(F_\omega^2)$. $\square$

Let F_8 be the graph obtained from K_ω and $K_{1,3}$ by adding an edge between a pendant vertex of $K_{1,3}$ and a vertex of K_ω, and F_9 be the graph obtained from F_8 by adding an edge between two pendant vertices (see Fig. 13).

Lemma 4.9. *Let F_ω^2 be the graph defined in Fig. 10. Then we have $\mu(F_\omega^2) > \mu(F_8) = \mu(F_9)$.*

Proof. It is easy to see that $\mu(F_8) = \mu(F_9)$ from Lemma 2.7. By direct calculation, we have

$$\begin{aligned}\Phi(F_8) &= x(x-\omega)^{\omega-2}[x^5 - (\omega+8)x^4 + (7\omega+22)x^3 \\ &\quad - (15\omega+25)x^2 + (10\omega+15)x - \omega - 4] \\ &= x(x-\omega)^{\omega-2}g_5(x).\end{aligned}$$

From Lemma 2.2, we have $\mu(F_8) > \omega + 1$. Then $\mu(F_8)$ is the largest root of the equation $g_5(x) = 0$.

From the proof of Lemma 4.3, we have $\mu(F_\omega^2)$ is the largest root of the equation

$$q_2(x) = x^5 - (\omega+9)x^4 + (8\omega+28)x^3 - (20\omega+38)x^2 + (16\omega+26)x - 2\omega - 8 = 0.$$

Since

$$\begin{aligned}(x-2)g_5(x) - (x-1)q_2(x) &= x^4 - (\omega+3)x^3 + (4\omega+1)x^2 - 3\omega x \\ &= x[x^3 - (\omega+3)x^2 + (4\omega+1)x - 3\omega]\end{aligned}$$

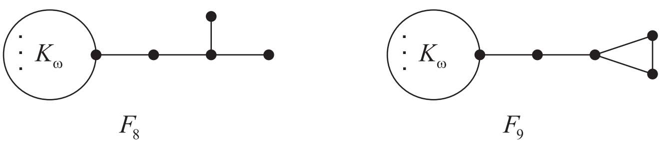

Fig. 13. Graphs F_8 and F_9.

$$\begin{aligned} &> x^2[x^2-(\omega+3)x+4\omega-2] \\ &= x^2[(x-3)(x-\omega)+\omega-2] > 0, \text{ when } x > \omega+1, \end{aligned}$$

we have $\mu(F_\omega^2) > \mu(F_8)$. $\square$

Theorem 4.10. *Among all connected graphs on $n \geq 9$ vertices with clique number $\omega = n-4$, the first three smallest Laplacian spectral radii are obtained from F_ω^1 $(= PK_{4,\omega})$; F_8, F_9; and F_ω^2, respectively.*

Proof. Let K_ω be the clique of G. From Lemma 4.7 and the proof of Case 1 of Theorem 3.7, we may assume that there exists exactly one vertex u outside of K_ω such that u is adjacent to some vertices of K_ω. We deal with the following three cases:

Case 1. If u is adjacent to t $(t \geq 2)$ vertices of K_ω, then from Lemmas 2.1 and 2.6, we have $\mu(G) \geq \mu(F_\omega^t)$. Moreover, the equality holds if and only if $G = F_\omega^t$. From Lemma 4.3, we have $\mu(F_\omega^t) > \mu(F_\omega^2)$ for $t \geq 3$. Thus we have $\mu(G) \geq \mu(F_\omega^2)$. Moreover, the equality holds if and only if $G = F_\omega^2$.

Case 2. If u is adjacent to exactly one vertex of K_ω and $d(u) \geq 3$, then from Lemmas 2.1 and 2.6, we have $\mu(G) \geq \mu(F_7)$. Thus from Lemma 4.8, we have $\mu(G) \geq \mu(F_7) > \mu(F_\omega^2)$.

Case 3. If u is adjacent to exactly one vertex of K_ω and $d(u) = 2$, then $G \in \{F_\omega^1, F_8, F_9\}$. By Lemmas 2.6 and 4.9, we have $\mu(F_\omega^2) > \mu(F_9) = \mu(F_8) > \mu(F_\omega^1)$.

The proof is complete. $\square$

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, MacMillan, New York, 1976.
[2] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1980.
[3] K.C. Das, An improved upper bound for Laplacian graph eigenvalues, Linear Algebra Appl. 368 (2003) 269–278.
[4] J.-M. Guo, On the second largest Laplacian eigenvalue of trees, Linear Algebra Appl. 404 (2005) 251–261.
[5] J.-M. Guo, The effect on the Laplacian spectral radius of a graph by adding or grafting edges, Linear Algebra Appl. 413 (2006) 59–71.
[6] J.-M. Guo, The Laplacian spectral radius of a graph under perturbation, Comput. Math. Appl. 54 (2007) 709–720.
[7] I. Gutman, D. Vidović, The largest eigenvalue of adjacency and Laplacian matrices, and ionization potentials of alkanes, Indian J. Chem. 41A (2002) 893–896.
[8] I. Gutman, D. Vidović, D. Stevanović, Chemical applications of the Laplacian spectrum. VI. On the largest Laplacian eigenvalue of alkanes, J. Serb. Chem. Soc. 67 (6) (2002) 407–413.
[9] J. Li, W.C. Shiu, W.H. Chan, On the Laplacian spectral radii of bipartite graphs, Linear Algebra Appl. 435 (2011) 2183–2192.
[10] J. Li, W.C. Shiu, W.H. Chan, The Laplacian spectral radius of some graphs, Linear Algebra Appl. 431 (2009) 99–103.
[11] R. Grone, R. Merris, The Laplacian spectrum of graph II, SIAM J. Discrete Math. 7 (1994) 221–229.
[12] R. Merris, Laplacian graph eigenvectors, Linear Algebra Appl. 278 (1998) 221–236.
[13] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197 198 (1994) 143–176.
[14] B. Mohar, S. Poljak, Eigenvalues in combinatorial optimization, in: R.A. Brualdi, S. Friedl, V. Klee (Eds.), Combinatorial and Graph-Theoretical Problems in Linear Algebra, IMA Volumes in Mathematics and its Applications, vol. 50, Springer-Verlag, 1993, pp. 107–151.
[15] J.-Y. Shao, J.-M. Guo, H.-Y. Shan, The ordering of trees and connected graphs by their algebraic connectivity, Linear Algebra Appl. 428 (2008) 1421–1438.
[16] J.-Y. Shao, L.-H. Shen, J.-M. Guo, Ordering of trees with smallest Laplacian spectral radii, J. Tongji Univ. (Nat. Sci.) 35 (2007) 552–555.
[17] L.-H. Shen, J.-Y. Shao, J.-M. Guo, The ordering of the connected graphs with the smallest Laplacian spectral radii, Chinese Ann. Math. Ser. A 29 (2) (2008) 273–282.
[18] P. Solé, Expanding and forwarding, Discrete Appl. Math. 58 (1995) 67–78.
[19] D. Stevanović, P. Hansen, The minimum spectral radius of graphs with a given clique number, Electron. J. Linear Algebra 17 (2008) 110–117.
[20] X.Y. Yuan, B.F. Wu, E.L. Xiao, The modifications of trees and the Laplacian spectrum, J. East China Norm. Univ. Natur. Sci. Ed. 2 (2004) 13–18.